LECTURE 16

Related Rates Continued

Continuing with the example last class, we restate the problem.

Example 1. A 15-foot ladder is resting against the wall. The bottom is initially10 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing?

Solution. The ladder, vertical wall and the floor form a right triangle. The length relationship here is via Pythagoras. Denote floor length and wall height $x(t)$ and $y(t)$ respectively. Then, we see that

$$x^2 + y^2 = 15^2$$

since the hypotenuse is always the same ladder, and thus of length 15. Note that both x and y are actually $x(t)$ and $y(t)$, functions of time.

Now, we want to know how fast the ladder is moving up the wall 12 seconds after we start pushing, so we want the rate of change of "wall height" $y'(12)$. Taking a derivative, we have

$$2xx' + 2yy' = 0 \implies x(t)\,x'(t) + y(t)\,y'(t) = 0.$$

We want to evaluate $\boxed{y'(12)}$. We identify what we know here already

$$\begin{aligned} x(12) &= 10 - \frac{1}{4} \times 12 = 7. \\ x'(12) &= -\frac{1}{4} \\ y(12) &= \sqrt{15^2 - 7^2} = \sqrt{176} = 4\sqrt{11}. \end{aligned}$$

Plugging back, we find

$$7 \times \left(-\frac{1}{4}\right) + 4\sqrt{11}y'(12) = 0 \implies y'(12) = \frac{7}{4}\frac{1}{4\sqrt{11}} = \frac{7}{16\sqrt{11}} \approx 0.1319\, ft/sec$$

Example 2. (Involving angles and trigonometry) Two people are standing 50 ft apart on the x-axis. One starts walking north at a rate such that the angle between them is changing at a rate of 0.01 rad/min, while the other person stays put. At what rate is distance between the two people changing when $\theta = 0.5$ radians?

Solution. The distance from the stationary person to the starting point of the other person is fixed at 50 ft. The distance between them at time t can be labeled as $z(t)$, the length of the hypotenuse. Then, we have the relationship

$$\cos(\theta(t)) = \frac{50}{z(t)} \implies \sec(\theta(t)) = \frac{z(t)}{50}.$$

We want $\frac{dz}{dt}\mid_{\theta=0.5}$. Taking a derivative with respect to t, we have

$$\begin{aligned} \frac{d}{dt}(\sec(\theta(t))) &= \frac{1}{50}\frac{d}{dt}z(t) \\ \implies \sec(\theta(t))\tan(\theta(t))\frac{d\theta}{dt} &= \frac{1}{50}\frac{dz}{dt} \end{aligned}$$

So, $\theta = 0.5$ can be plugged in here, while also we know $\frac{d\theta}{dt} = 0.01$ is a constant (as given). It should be positive because the angle is always increasing. Thus,

$$\frac{dz}{dt} = 50\sec(0.5)\tan(0.5)(0.01) \approx 0.311\, ft/min.$$

Derivatives of inverse Functions and Logarithms

Now, switching back to just doing derivatives, we now consider first the derivative of inverse functions.

Motivation of Inverse Functions

Why are inverse functions useful? Have you ever encountered them yet you did not notice? In fact, when you are solving a quadratic equation, you used the notion of inverses already. Suppose we look for the zeros of the function

$$f(x) = x^2 + 3x + 2.$$

For those who are keen, you will notice right away $x = -2$ and $x = -1$ are the zeros. If you can't factor quickly, then the quadratic formula will also give you the answer,

$$x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 2}}{2}.$$

Note that your original equation is quadratic, and now the formula that gives the root involves square roots. You definitely know that to undo a square, you take the square root, making these two operations inverses of each other. So, a coincidence here for the quadratic formula?

What if the question is harder,

$$g(x) = x^3 + 5x + 1?$$

You are specifically looking for the x^*'s such that $g(x^*) = 0$. The function inverse undoes the original function, that is,

$$g^{-1}(g(x)) = x.$$

Therefore, if you know the function inverse $g^{-1}(x)$, then to find the zeros of $g(x)$, you do

$$g^{-1}(g(x^*)) = g^{-1}(0) \implies x^* = g^{-1}(0)$$

which means you just need to evaluate at $x = 0$ for the function $g^{-1}(x)$ to get the zeros of $g(x)$.

Derivative of an Inverse Function

Let's consider a simple function.

$$f(x) = \frac{1}{2}x + 1,$$

where we find the inverse to be

$$f^{-1}(x) = 2x - 2.$$

(If you don't know how to find the inverse of a function, you should learn the following quickly. Take your original function $y = \frac{1}{2}x + 1$, swap x and y and solve for y, i.e. $x = \frac{1}{2}y + 1 \implies y = 2x - 2$. Now, you call this $y = f^{-1}(x)$. Same process for harder looking functions.)

Let's find the derivative of both functions.

$$f'(x) = \frac{1}{2}, \quad \left(f^{-1}(x)\right)' = 2.$$

Hmm, reciprocal. Is this a coincidence?

Consider a differentiable function $f(x)$ and a point on it $(a, f(a))$. The inverse $f^{-1}(x)$ will take this point to $(f(a), a)$. The slope at $x = a$ for $f(x)$ is $f'(a)$. We just "learned" that the slope at the reflected point should be the reciprocal, that is, by writing $b = f(a)$,

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

Great, this is getting more general, though for a specific point $(a, f(a))$. We want to show if this formula holds for all points b, given that f is a differentiable function.

Theorem. *If f is differentiable on some interval I, and f' is never zero on I, then f^{-1} is differentiable everywhere on its domain (the range of f). Furthermore, given a point b in the domain of f^{-1}, we have*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Proof. The first claim is nontrivial to prove (but easy to visualise – when you flip a differentiable function about the origin, you won't lose differentiability unless the original slope is 0 so now you have a vertical slope).

We prove the second claim, the formula. We rely on the definition of the inverse function. By the chain rule,

$$\begin{aligned} f\left(f^{-1}(x)\right) &= x \\ \implies \frac{d}{dx}\left(f\left(f^{-1}(x)\right)\right) &= 1 \\ \implies f'\left(f^{-1}(x)\right) \cdot \left(\frac{d}{dx} f^{-1}(x)\right) &= 1 \\ \implies \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'\left(f^{-1}(x)\right)}. \end{aligned}$$

□

Let's test our theory on functions with known inverses.

Example. Consider $f(x) = x^2$ for $x > 0$ and thus $f'(x) = 2x$. We know its inverse $f^{-1}(x) = \sqrt{x}$ for $x > 0$, and thus also by power rule, $\left(f^{-1}\right)'(x) = \frac{1}{2\sqrt{x}}$. Let's see if the above formula works.

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{2f^{-1}(x)} = \frac{1}{2\sqrt{x}},$$

works!

A great advantage of the formula is that you don't have to find the explicit form of the inverse to find its derivative at a point.

Example. Let $f(x) = x^3 - 2$, $x > 0$. Find the value of $\frac{df^{-1}}{dx}$ at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution. We want $\left(f^{-1}\right)'(6)$. By the theorem, it is equal to

$$\left(f^{-1}\right)'(6) = \frac{1}{f'\left(f^{-1}(6)\right)}.$$

So, we go on to find the two ingredients. First, $f^{-1}(6) = 2$. Second $f'(x) = 3x^2 \implies f'(2) = 3 \cdot (2)^2 = 12$. Therefore,

$$\left(f^{-1}\right)'(6) = \frac{1}{f'\left(f^{-1}(6)\right)} = \frac{1}{12}.$$